BALANCING WITH FIBONACCI POWERS

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Abstract. The Diophantine equation $F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l$ in positive integers $n,\ r,\ k,\ l$ with $n \geq 2$ is studied where F_n is the n^{th} term of the Fibonacci sequence.

1. Introduction

As usual $\{F_n\}_{n=0}^{\infty}$ denotes the sequence of Fibonacci numbers and $\{L_n\}_{n=0}^{\infty}$ the sequence of Lucas numbers. It is well known that the recurrence relations of these two sequences are

$$F_0 = 0,\ F_1 = 1 \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n,$$
$$L_0 = 2,\ L_1 = 1 \qquad \text{and} \qquad L_{n+2} = L_{n+1} + L_n,$$

and their Binet forms are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n, \tag{1.1}$$

respectively, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. In the sequel, we investigate the Diophantine equation

$$F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l \tag{1.2}$$

in positive integers $n,\ r,\ k,\ l$ with $n \geq 2$. Panda [4] has treated the special case $k = l = 1$. The authors believe that the following conjecture is true.

Conjecture 1.1. *The only quadruple $(n, r, k, l) = (4, 3, 8, 2)$ of positive integers satisfy equation (1.2).*

We validate this claim to some extent by showing that several particular cases of (1.2) do not possess any solution.

2. Auxiliary results

The results presented in this section are required to establish certain claims on Conjecture 1.1.

The following are some identities on Fibonacci numbers.

Lemma 2.1. **(a)** $\sum_{k=1}^{n} F_k = F_{n+2} - 1$,

(b) $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$,

(c) $\sum_{k=1}^{n} F_k^3 = \frac{F_{3n+2} + 6\cdot(-1)^{n-1}F_{n-1} + 5}{10}$,

(d) $F_n \leq L_n$, *and equality holds if and only if* $n = 1$,

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(e) $F_{2n} = F_n(F_{n+1} + F_{n-1})$,

(f) $F_n^2 = \frac{L_{2n} - 2(-1)^n}{5}$,

(g) $F_n^3 = \frac{F_{3n} - 3(-1)^n F_n}{5}$.

Proof. The proofs of these statements are well-known. However, statements (a) to (d) can be proved, for instance, by induction (especially, (d) appears in [1]). The statements (e) to (g) can be verified using the Binet formulae for L_n and F_n given in (1.1). □

The following result, which is a part of Lemma 5 in [3], gives upper and lower bounds for Fibonacci numbers in terms of powers of α.

Lemma 2.2. *Let u_0 be a positive integer. For $i = 1, 2$ put $\delta_i = \log_\alpha \left(\left(1 + (-1)^{i-1} \left(\frac{|\beta|}{\alpha} \right)^{u_0} \right) / \sqrt{5} \right)$. Then for all integers $u \geq u_0$ inequalities $\alpha^{u+\delta_2} \leq F_u \leq \alpha^{u+\delta_1}$.*

In order to make the application of Lemma 2.2 more convenient, we take $u_0 \geq 6$ and get the following result.

Corollary 2.3. *If $u_0 \geq 6$, then $\delta_1 < -1.66$ and $\delta_2 > -1.68$.*

The following result, which is Lemma 6 in [3], gives upper bounds for linear combinations of powers of α and 1 in terms of powers of α.

Lemma 2.4. *Suppose that $a > 0$ and $b \geq 0$ are real numbers and u_0 is a positive real number. Then $a\alpha^u + b \leq \alpha^{u+\kappa}$ holds for any $u \geq u_0$ where $\kappa = \log_\alpha \left(a + \frac{b}{\alpha^{u_0}} \right)$.*

3. The results

In this section, we present some results to support Conjecture 1.1. The first result deals with the non-existence of solutions of (1.2) when $k \leq l$.

Theorem 3.1. *The Diophantine equation $F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l$ has no solution in positive integers n and r with $n \geq 2$ if $k \leq l$.*

Proof. For $k \leq l$, using

$$F_1 + F_2 + \cdots + F_{n-1} = F_{n+1} - 1$$

we get

$$F_1^k + F_2^k + \cdots + F_{n-1}^k \leq (F_1 + F_2 + \cdots + F_{n-1})^k = (F_{n+1} - 1)^k < F_{n+1}^k \leq F_{n+1}^l,$$

showing that (1.2) has no solution in the positive integers n and r with $n \geq 2$ if $k \leq l$. □

Even if $k > l$, it can be proved that many particular cases of (1.2) do not possess any solution. The following result ascertains this claim when $k = 2$ and $l = 1$.

Theorem 3.2. *The Diophantine equation $F_1^2 + F_2^2 + \cdots + F_{n-1}^2 = F_{n+1} + F_{n+2} + \cdots + F_{n+r}$ has no solution in positive integers n and r with $n \geq 2$.*

Proof. By virtue of Lemma 2.1(a) and (b), the equation

$$F_1^2 + F_2^2 + \cdots + F_{n-1}^2 = F_{n+1} + F_{n+2} + \cdots + F_{n+r}$$

is equivalent to

$$F_{n-1}F_n + F_{n+2} = F_{n+r+2}. \tag{3.1}$$

Since $F_{n-1}F_n + F_{n+2}$ is not a Fibonacci number when $n = 2,\ 3,\ \ldots,\ 6$, we can safely assume that $n \geq 7$. Using Corollary 2.3, we find the upper and lower bounds for both sides of (3.1). Firstly, we observe that

$$F_{n-1}F_n + F_{n+2} > F_{n-1}F_n > \alpha^{n-1.68}\alpha^{n-1-1.68} = \alpha^{2n-4.36}. \tag{3.2}$$

On the other hand,

$$F_{n-1}F_n + F_{n+2} < \alpha^{n-1.66}\alpha^{n-1-1.66} + \alpha^{n+2-1.66} = \alpha^{n+0.34}(\alpha^{n-4.66} + 1). \tag{3.3}$$

Using Lemma 2.4 with $a = b = 1$, we obtain $\kappa < 0.68$, and by virtue of (3.3),

$$F_{n-1}F_n + F_{n+2} < \alpha^{n+0.34}(\alpha^{n-4.66} + 1) < \alpha^{n+0.34}\alpha^{n-4.66+0.68} = \alpha^{2n-3.64}. \tag{3.4}$$

Again, by Corollary 2.3,

$$\alpha^{n+r+0.32} = \alpha^{n+r+2-1.68} < F_{n+r+2} < \alpha^{n+r+2-1.66} = \alpha^{n+r+0.34}. \tag{3.5}$$

Thus, if (3.1) holds for the positive integers n and r, then by virtue of (3.2), (3.4) and (3.5), we get

$$\max\{2n - 4.36\,,\ n + r + 0.32\} < \min\{2n - 3.64\,,\ n + r + 0.34\},$$

which yields the inequalities

$$2n - 4.36 < n + r + 0.34$$

and

$$n + r + 0.32 < 2n - 3.64.$$

Hence, the positive integers n and r satisfy

$$n - 4.7 < r < n - 3.96,$$

which yield $r = n - 4$. Thus (3.1) reduces to

$$F_{n-1}F_n + F_{n+2} = F_{2n-2}. \tag{3.6}$$

if $n \geq 7$. But, by Lemma 2.1(e), (3.6) simplifies to

$$F_{n+2} = F_{n-1}F_{n-2}. \tag{3.7}$$

It is easy to see that (3.7) is not true for $n = 8,\ 9,\ 10$. If $n > 10$, i.e., $n+2 > 12$ then by virtue of the primitive divisor theorem [2], F_{n+2} has a prime factor that does not divide any of F_{n-1} and F_{n-2}. Hence (3.7) is not satisfied for any $n \geq 7$ and therefore (3.1) has no solution. □

The following result ascertains that there is no solution to (1.2) when $k = 3$ and $l = 1$.

Theorem 3.3. *The Diophantine equation* $F_1^3 + F_2^3 + \cdots + F_{n-1}^3 = F_{n+1} + F_{n+2} + \cdots + F_{n+r}$ *has no solution in positive integers* n *and* r *with* $n \geq 2$.

Proof. By virtue of Lemma 2.1(a), the equation

$$F_1^3 + F_2^3 + \cdots + F_{n-1}^3 = F_{n+1} + F_{n+2} + \cdots + F_{n+r}$$

reduces to

$$F_1^3 + F_2^3 + \cdots + F_{n-1}^3 = F_{n+r+2} - F_{n+2}.$$

Since $F_1^3 + F_2^3 + \cdots + F_{n-1}^3 + F_{n+2}$ does not yield a Fibonacci number when $n = 2,\ 3,\ 4$, without loss of generality, we may assume that $n \geq 5$. Further, by Lemma 2.1(c), the last equation is equivalent to

$$F_{3n+2} + 10F_{n+2} + 6(-1)^{n-1}F_{n-1} + 5 = 10F_{n+r+2}. \tag{3.8}$$

We apply Corollary 2.3 and get the upper and lower bounds for both sides of (3.8) as follows:

$$F_{3n+2} + 10F_{n+2} + 6(-1)^{n-1}F_{n-1} + 5 > F_{3n+2} > \alpha^{3n+2-1.68} = \alpha^{3n+0.32}, \tag{3.9}$$

while

$$F_{3n+2} + 10F_{n+2} + 6(-1)^{n-1}F_{n-1} + 5 < F_{3n+2} + 21F_{n+2} < \alpha^{3n+2-1.66} + 21\alpha^{n+2-1.66}. \tag{3.10}$$

Since $\log_\alpha 21 < 6.37$, we obtain

$$\alpha^{3n+2-1.66} + 21\alpha^{n+2-1.66} < \alpha^{3n+0.34} + \alpha^{n+6.71} = \alpha^{n+6.71}(\alpha^{2n-6.37} + 1). \tag{3.11}$$

Now $n \geq 5$ entails $2n - 6.37 > 3$. By Lemma 2.4 with $a = b = 1$, we obtain $\kappa < 0.45$ and subsequently, we have

$$\alpha^{n+6.71}(\alpha^{2n-6.37} + 1) < \alpha^{n+6.71}\alpha^{2n-6.37+0.45} = \alpha^{3n+0.79}. \tag{3.12}$$

Using (3.9), (3.10), (3.11) and (3.12) we get

$$\alpha^{3n+0.32} < F_{3n+2} + 10F_{n+2} + 6(-1)^{n-1}F_{n-1} + 5 < \alpha^{3n+0.79}. \tag{3.13}$$

Similarly, since $4.78 < \log_\alpha 10 < 4.79$, we get

$$10F_{n+r+2} > \alpha^{4.78}\alpha^{n+r+2-1.68} = \alpha^{n+r+5.1} \tag{3.14}$$

and

$$10F_{n+r+2} < \alpha^{4.79}\alpha^{n+r+2-1.66} = \alpha^{n+r+5.13}. \tag{3.15}$$

We now combine (3.14) and (3.15) and get

$$\alpha^{n+r+5.1} < 10F_{n+r+2} < \alpha^{n+r+5.13}. \tag{3.16}$$

In view of (3.8), (3.13) and (3.16), we have the system of inequalities

$$n + r + 5.1 < 3n + 0.79$$

and

$$3n + 0.32 < n + r + 5.13,$$

yielding

$$2n - 4.81 < r < 2n - 4.31,$$

which is impossible since n and r are integers. □

Equation (1.2) does not exhibit any solution even if $k = 3$ and $l = 2$. The following result ascertains this fact.

Theorem 3.4. *The Diophantine equation* $F_1^3 + F_2^3 + \cdots + F_{n-1}^3 = F_{n+1}^2 + F_{n+2}^2 + \cdots + F_{n+r}^2$ *has no solution in positive integers* n *and* r *with* $n \geq 2$.

Proof. Application of Lemma 2.1(b) and (c) converts the equation

$$F_1^3 + F_2^3 + \cdots + F_{n-1}^3 = F_{n+1}^2 + F_{n+2}^2 + \cdots + F_{n+r}^2$$

to

$$F_{3n+2} + 10F_nF_{n+1} + 6(-1)^{n-1}F_{n-1} + 5 = 10F_{n+r}F_{n+r+1}. \tag{3.17}$$

It is easy to check that the above equation has no solution if $n = 2,\ 3,\ \ldots,\ 6$. Supposing $n \geq 7$, observing that $4.78 < \log_\alpha 10 < 4.79$, and using Lemma 2.4 and Corollary 2.3 we find

$$\alpha^{2n+2r+2.42} < 10F_{n+r}F_{n+r+1} < \alpha^{2n+2r+2.47}.$$

On the other hand, by (3.9)

$$F_{3n+2} + 10F_nF_{n+1} + 6(-1)^{n-1}F_{n-1} + 5 > F_{3n+2} > \alpha^{3n+0.32},$$

while

$$F_{3n+2} + 10F_nF_{n+1} + 6(-1)^{n-1}F_{n-1} + 5 < F_{3n+2} + 21F_{n+1}^2 <$$
$$< \alpha^{3n+2-1.66} + \alpha^{6.37}\alpha^{2(n+1-1.66)} = \alpha^{2n+5.05}(\alpha^{n-4.71} + 1).$$

Now $n - 4.71 > 3$, and by Lemma 2.4 with $a = b = 1$, we have $\kappa < 0.68$ and hence

$$\alpha^{2n+5.05}(\alpha^{n-4.71} + 1) < \alpha^{3n+1.02}.$$

Comparing the upper and lower bounds of both sides of (3.17), we arrive at the system of inequalities

$$2n + 2r + 2.42 < 3n + 1.02$$

and

$$3n + 0.32 < 2n + 2r + 2.47;$$

the last two inequalities imply

$$2r + 1.4 < n < 2r + 2.15.$$

Thus $n = 2r + 2$, and our problem reduces to proving that for no positive integer r, the equation

$$F_1^3 + F_2^3 + \cdots + F_{2r+1}^3 = F_{2r+3}^2 + F_{2r+4}^2 + \cdots + F_{3r+2}^2$$

is satisfied. For this, it is sufficient to show that for every positive integer r,

$$F_1^3 + F_2^3 + \cdots + F_{2r+1}^3 < F_{2r+3}^2 + F_{2r+4}^2 + \cdots + F_{3r+2}^2. \tag{3.18}$$

We prove (3.18) by induction. Since

$$F_1^3 + F_2^3 + F_3^3 = 10 < 25 = F_5^2,$$

it is sufficient to prove that

$$F_{2r+2}^3 + F_{2r+3}^3 < F_{3r+3}^2 + F_{3r+4}^2 + F_{3r+5}^2 - F_{2r+3}^2 - F_{2r+4}^2;$$

by Lemma 2.1(d), (f) and (g), the last inequality is equivalent to

$$F_{6r+9} + F_{6r+6} + 3F_{2r+1} < L_{6r+10} + L_{6r+8} + L_{6r+6} - L_{4r+8} - L_{4r+6} \pm 2. \tag{3.19}$$

Clearly, the combination of $F_{6r+6} < L_{6r+6}$,

$$F_{6r+9} < L_{6r+9} = L_{6r+10} - L_{6r+8} < L_{6r+10} - L_{4r+8}$$

and

$$3F_{2r+1} < F_{2r+5} < L_{2r+5} < L_{6r+6} = L_{6r+7} - L_{6r+5} <$$
$$< L_{6r+7} - L_{4r+6} < L_{6r+8} - L_{4r+6} - 2$$

justifies (3.19). □

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