ON GENERALIZED BALANCING SEQUENCES

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Abstract. Let $R_i = R(A, B, R_0, R_1)$ be a second order linear recurrence sequence. In the present paper we prove that any sequence $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0, (A, B) \neq (0, 1)$ is not a balancing sequence.

1. Introduction

In 1999, A. Behera and G. K. Panda [3] defined the notion of balancing numbers. A positive integer n is called a *balancing number* if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+k)$$

for some $k \in \mathbb{N}$. Then k is called the balancer of n. It is easy to see that 6, 35, and 204 are balancing numbers with balancers 2, 14, and 84, respectively. In [3] the authors proved that balancing numbers fulfill the following recurrence relation

$$B_{n+1} = 6B_n - B_{n-1} \quad (n > 1),$$

where $B_0 = 1$ and $B_1 = 6$. In [5], R. Finkelstein studied "The house problem" and introduced the notion of first-power numerical center which coincides with the notion of balancing numbers except for the number 1 which is a first-power numerical center but not a balancing number.

In [8], the authors defined the notion of (k, l)*-power numerical center* or (k, l)*-balancing number*. More precisely let y, k, l be fixed positive integers with $y > 1$. We call the positive integer $x, (x \leq y)$, a (k, l)*-power numerical center* or (k, l)*-balancing number* for y if

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

In [5], R. Finkelstein proved that there are no second-power numerical centers (in this case $k = l = 2$). Later on R. Steiner [13], proved that there are no third-power numerical centers (in this case $k = l = 3$). (Here we mention that R. Finkelstein and R. Steiner is the same person.) In the case $k = 4$ and $k = 5$ he conjectured a negative answer. Later on P. Ingram in [6] using the explicit lower bounds on linear forms in elliptic logarithms, proved that there are no nontrivial fifth-power numerical centers. In the same paper he proved that there are only finitely many nth power numerical centers.

K. Liptai, F. Luca, Á. Pintér, and L. Szalay [8] obtained certain effective and ineffective finiteness theorems for (k, l) numerical centers. Their results are based on Baker's theory and a result of Cs. Rakaczki [11], respectively. Furthermore, they proved that there exists no (k, l) numerical center with $l > k$.

The research was supported in part by the Hungarian Academy of Sciences (A.B.), and by grants T67580 (A.B.) and T75566 (A.B., I.P.) of the Hungarian National Foundation for Scientific Research, and the János Bolyai Research Scholarship (A.B.) and TT SK-8/2008 (K. L.).

In [7], K. Liptai searched for those balancing numbers which are Fibonacci numbers, too. Using the results of A. Baker and G. Wüstholz [2] he proved that there are no Fibonacci balancing numbers. Using another method L. Szalay [14] proved that there are no Lucas balancing numbers.

Later G. K. Panda and P. K. Ray [9] slightly modified the definition of a balancing number and introduced the notion of a cobalancing number. A positive integer n is called a *cobalancing number* if

$$1 + 2 + \cdots + (n-1) + n = (n+1) + (n+2) + \cdots + (n+K)$$

for some $K \in \mathbb{N}$. In this case K is called the cobalancer of n.

They also proved that the cobalancing numbers fulfill the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad (n > 1),$$

where $b_0 = 1$ and $b_1 = 6$. Moreover they found that every balancer is a cobalancing number and every cobalancer is a balancing number.

In [10], G. K. Panda gave another possible generalization of balancing numbers. Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of real numbers. We call an element a_n of this sequence a *sequence-balancing* number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + a_{n+2} + \cdots + a_{n+k}$$

for some $k \in \mathbb{N}$. Similarly, one can define the notion of *sequence cobalancing numbers*. In [10] it was proved that there does not exist any sequence balancing number in the Fibonacci sequence. The sequence $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence sequence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2} \ (i \geq 2)$$

holds, where $A, B \neq 0, R_0, R_1$ are fixed rational integers and $|R_0| + |R_1| > 0$. The polynomial $f(x) = x^2 - Ax - B$ is called the companion polynomial of the sequence $R = R(A, B, R_0, R_1)$. Let $D = A^2 + 4B$ be the discriminant of f. The roots of the companion polynomial will be denoted by α and β. Using this notation if $D \neq 0$, as it is well-known, we may write

$$R_i = \frac{a\alpha^i - b\beta^i}{\alpha - \beta}$$

for $i \geq 2$, where $a = R_1 - R_0\beta$ and $b = R_1 - R_0\alpha$.

As a generalization of the notion of a balancing number, we will call a binary recurrence $R_i = R(A, B, R_0, R_1)$ a *balancing sequence* if

$$R_1 + R_2 + \cdots + R_{n-1} = R_{n+1} + R_{n+2} + \cdots + R_{n+k} \tag{1}$$

holds for some $k \geq 1$ and $n \geq 2$.

In the present paper we prove that any sequence $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence.

2. Results

Theorem 1. *There is no balancing sequence of the form $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0$ except for $(A, B) = (0, 1)$ in which case (1) has infinitely many solutions $(n, k) = (n, n-1)$ and $(n, k) = (n, n)$ for $n \geq 2$.*

As a consequence of Theorem 1 above, we consider the question of Lucas-sequences. As it is well-known, a sequence

$$R_i = R(A, B, 0, 1) = \frac{\alpha^i - \beta^i}{\alpha - \beta}$$

is called a Lucas-sequence, if $\frac{\alpha}{\beta}$ is not a root of unity and $\gcd(A, B) = 1$.

Corollary 1. *Let $R_i = R(A, B, 0, 1)$ be a Lucas-sequence with $A^2 + 4B > 0$. Then R_i is not a balancing sequence.*

3. Auxiliary Results

Lemma 1. *Let $n \geq 2$ and $k \geq 1$ be integers and consider the function $F : \mathbb{R} \setminus \{1\} \to \mathbb{R}$,*

$$F(x) = \frac{x^{n+k+1} - x^{n+1} - x^n + x}{x - 1}.$$

Then F is strictly increasing on the interval $(-\infty, -1]$ if $n + k$ is odd and F is strictly decreasing on the interval $(-\infty, -1]$ if $n + k$ is even.

Proof. The derivative F' of F is

$$F'(x) = \frac{(n + k)x^{n+k+1} - (n + k + 1)x^{n+k} - nx^{n+1} + 2x^n + nx^{n-1} - 1}{(x - 1)^2}.$$

We may suppose that $x \leq -1$. Hence we have $x = -|x|$. Therefore $F'(x)$ can be rewritten in the form

$$F'(x) = \frac{|x|^{n+k}g(x) - |x|^{n+1}h(x)}{(x - 1)^2},$$

where

$$g(x) = (-1)^{n+k}(-n-k-1-(n+k)|x|) \text{ and } h(x) = n(-1)^{n+1} - \frac{2(-1)^n}{|x|} - \frac{n(-1)^{n-1}}{|x|^2} + \frac{1}{|x|^{n+1}}. \tag{2}$$

Now, if $n + k$ is odd, then since $|x| \geq 1$ and since $n + 1$ and $n - 1$ have the same parity, by (2) one gets

$$g(x) \geq 2n + 2k + 1 \text{ and } h(x) < n + 2 + 1 = n + 3.$$

Hence,

$$F'(x) > \frac{(n + 2k - 2)|x|^{n+1}}{(x - 1)^2},$$

so for $k \geq 1$ and $n \geq 2$ this leads to $F'(x) > 0$ for $x \leq -1$ and the lemma follows.

Finally, if $n + k$ is even and $|x| \geq 1$ we have

$$g(x) \leq -2n - 2k - 1 \text{ and } h(x) > -n - 2 + 1 = -n - 1,$$

so for $k \geq 1$ we get

$$F'(x) < \frac{(-n - 2k)|x|^{n+1}}{(x - 1)^2}.$$

Since $n \geq 2$ and $k \geq 1$ one observes that $F'(x) < 0$ holds for $x \leq -1$, so the lemma follows. □

4. Proof of Theorem 1

Consider the sequence $R_i = R(A, B, 0, R_1)$ with $R_1 \neq 0$, companion polynomial $f(x) = x^2 - Ax - B$, and positive discriminant $D = A^2 + 4B > 0$. Since $R(A, B, 0, R_1) = R_1 \cdot R(A, B, 0, 1)$ one can observe that $R(A, B, 0, R_1)$ is a balancing sequence (i.e (1) holds) if and only if $R(A, B, 0, 1)$ is a balancing sequence. Thus we may assume that $R_1 = 1$ that is, in what follows we may deal without loss of generality with the sequence $R_i = R(A, B, 0, 1)$.

We distinguish several subcases according to $A = 0$ or to the signs of A and B, respectively.

Case 1: $A = 0$.

Since $0 < D = A^2 + 4B$ it follows that $B > 0$. The roots of the companion polynomial $f(x) = x^2 - B$ are $\alpha = \sqrt{B}$ and $\beta = -\alpha = -\sqrt{B}$. Thus we have the sequence

$$R_i = \frac{\sqrt{B}^i - (-\sqrt{B})^i}{2\sqrt{B}}, \; i \geq 0.$$

Now, if $B = 1$ then R_i is of the form

$$R_i = \frac{1^i - (-1)^i}{2}, \; i \geq 0$$

which is obviously a balancing sequence. Further, the resulting equation (1) in this case has infinitely many solutions (n, k), namely $(n, k) = (n, n - 1)$ and (n, n) for $n \geq 2$.

If $B > 1$ then for $i \geq 0$ we have

$$R_i = \begin{cases} 0, & \text{if } i \text{ is even,} \\ B^{\frac{i-1}{2}}, & \text{if } i \text{ is odd.} \end{cases}$$

Suppose that (1) holds with $n \geq 2$ odd. Since in this case $R_{n-1} = R_{n+1} = 0$ and the left hand side of (1) is $\frac{B^{\frac{n-1}{2}}-1}{B-1}$, we may obviously assume that $k \geq 2$. Now, for the right hand side of (1) we have

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} \geq R_{n+2} = B^{\frac{n+1}{2}}.$$

But this leads to a contradiction in view of (1), $B > 1$, $n \geq 2$ and

$$\frac{B^{\frac{n-1}{2}} - 1}{B - 1} < B^{\frac{n+1}{2}}.$$

Finally, if equation (1) holds with $n \geq 2$ even then $R_{n-1} = B^{\frac{n-2}{2}}$ and $R_{n+1} = B^{\frac{n}{2}}$. Hence the left hand side of (1) is $\frac{B^{\frac{n}{2}}-1}{B-1}$ while for the right hand side we have the lower bound $B^{\frac{n}{2}}$. This is impossible by (1), $B > 1$, $n \geq 2$ and

$$\frac{B^{\frac{n}{2}} - 1}{B - 1} < B^{\frac{n}{2}}.$$

Hence, in this case there is no balancing sequence apart from $B = 1$.

Case 2: $A > 0$.

Let α and β be the roots of the companion polynomial $f(x) = x^2 - Ax - B$. One observes that f has a dominant root which we will denote by α. (Note that α is a dominant root of f if $|\alpha| > |\beta|$). In this case we have

$$\alpha = \frac{A + \sqrt{A^2 + 4B}}{2}, \; \beta = \frac{A - \sqrt{A^2 + 4B}}{2}.$$

Since $A \geq 1$ and $D = A^2 + 4B > 0$ we obviously have $\alpha > 1$. Further, since $R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta}$ $(i \geq 0)$ and $|\beta| < \alpha$ we get that $R_i > 0$ for $i \geq 1$. Suppose that (1) holds for some $n \geq 2$ and $k \geq 1$. We derive an upper bound for the left hand side of (1). Since

$$R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta} < \frac{2\alpha^i}{\alpha - \beta},$$

we get

$$R_1 + R_2 + \cdots + R_{n-1} < \frac{2}{\alpha - \beta} \sum_{i=1}^{n-1} \alpha^i = \left(\frac{2\alpha}{\alpha - \beta}\right)\left(\frac{\alpha^{n-1} - 1}{\alpha - 1}\right). \tag{3}$$

Further, since $R_i > 0$ for all $i \geq 1$ we get for the right hand side of (1) the lower bound

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} \geq \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Suppose first that $\beta^{n+1} < 0$. Then

$$R_{n+1} + R_{n+2} + \cdots + R_{n+k} > \frac{\alpha^{n+1}}{\alpha - \beta}. \tag{4}$$

Further, we see that $\beta^{n+1} < 0$ holds if and only if $\beta < 0$ (and $n + 1$ odd). Hence, we may assume that $B > 0$. Now, by (1), (3), and (4) we obtain

$$\frac{\alpha^{n+1}}{\alpha - \beta} < \left(\frac{2\alpha}{\alpha - \beta}\right)\left(\frac{\alpha^{n-1} - 1}{\alpha - 1}\right),$$

which leads to

$$\alpha^2 - \alpha - 2 < -\frac{2}{\alpha^{n-1}}. \tag{5}$$

Thus (5) implies that $\alpha = \frac{A + \sqrt{A^2 + 4B}}{2} < 2$ and since $A > 0$ and $B > 0$ this can occur only if $A = B = 1$. In this case the resulting sequence is the Fibonacci sequence and for it

$$R_1 + \cdots + R_{n-1} = F_1 + \cdots + F_n = F_{n+1} - 1 < F_{n+1} = R_{n+1}. \tag{6}$$

Thus (6) shows that there is no balancing sequence if $\beta^{n+1} < 0$.

Suppose now that $\beta^{n+1} > 0$ and assume that (1) holds for some $n \geq 2$ and $k \geq 1$. In this case the upper bound (3) for the left-hand side of (1) remains valid. Since $\alpha > |\beta|$ for R_{n+1} we get the following lower bound

$$R_{n+1} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\alpha^{n+1} - |\beta|^{n+1}}{\alpha - \beta} = \frac{(\alpha - |\beta|)(\alpha^n + \cdots + |\beta|^n)}{\alpha - \beta} > \frac{\Delta\alpha^n}{\alpha - \beta}, \tag{7}$$

where $\Delta = \alpha - |\beta| = \sqrt{A^2 + 4B}$. Hence, using (1), (3), and (7) we get

$$\frac{\Delta\alpha^n}{\alpha - \beta} < \left(\frac{2\alpha}{\alpha - \beta}\right)\left(\frac{\alpha^{n-1} - 1}{\alpha - 1}\right)$$

which leads to

$$\Delta\alpha^{n+1} - (\Delta+2)\alpha^n < -2\alpha. \tag{8}$$

But (8) is a contradiction if $\alpha \geq \frac{\Delta+2}{\Delta} = 1 + \frac{2}{\Delta}$. Finally, if $\alpha < 1 + \frac{2}{\Delta}$ then since $\Delta \geq 1$ we get that $\alpha < 3$. Thus, those values of the pair (A, B) for which $\alpha = \frac{A+\sqrt{A^2+4B}}{2} < 3$ and $A^2 + 4B > 0$ are the following

$$(A, B) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (3, -2), (3, -1)\}.$$

Now, if $(A, B) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (3, -1)\}$ we see that $\Delta \geq \sqrt{5}$ and hence

$$\alpha < 1 + \frac{2}{\sqrt{5}}, \tag{9}$$

which implies that the only value of $\alpha = \frac{A+\sqrt{A^2+4B}}{2}$ for which (9) holds is $A = B = 1$, i.e. $\alpha = \frac{1+\sqrt{5}}{2}$. But in this case the resulting sequence is again the Fibonacci sequence for which we have already checked that (1) cannot hold.

Finally, if $(A, B) = (3, -2)$ the resulting sequence is $R_i = 2^i - 1$ for $i \geq 0$. Assume that (1) holds for this sequence. One can easily see that the left hand side of (1) in this case is $2^n - n - 1$. Further, for the right hand side of (1) we have the lower bound $R_{n+1} = 2^{n+1} - 1$. But for $n \geq 2$

$$2^{n+1} - 1 > 2^n - n - 1$$

which shows that the sequence $R_i = 2^i - 1$ cannot be a balancing sequence. So there is no balancing sequence with $A > 0$ and $D = A^2 + 4B > 0$.

Case 3: $A < 0$ and $B < 0$.

We work as in the previous case. Let α denote the dominant root of the companion polynomial $f(x) = x^2 - Ax - B$. Since $A < 0$ and $A^2 + 4B > 0$ we have

$$\alpha = \frac{A - \sqrt{A^2+4B}}{2} \text{ and } \beta = \frac{A + \sqrt{A^2+4B}}{2}.$$

Since $-B = \alpha\beta$ we see that $\beta < 0$. Now, if R_i is a balancing sequence then by (1) and $R_i = \frac{\alpha^i - \beta^i}{\alpha - \beta}$ we get

$$\frac{\alpha^{n+k+1} - \alpha^{n+1} - \alpha^n + \alpha}{\alpha - 1} = \frac{\beta^{n+k+1} - \beta^{n+1} - \beta^n + \beta}{\beta - 1}, \tag{10}$$

where $n \geq 2$ and $k \geq 1$. Thus from (10) we have

$$F(\alpha) = F(\beta), \tag{11}$$

where F is the function defined in Section 3. Now, if $\beta \leq -1$ then by $\alpha < \beta$ we get by Lemma 1 that $F(\alpha) < F(\beta)$ if $n + k$ is odd and $F(\alpha) > F(\beta)$ if $n + k$ is even. But this contradicts (11). Hence, we may assume that $-1 < \beta < 0$ and we may suppose without loss of generality that $\alpha \leq \alpha_0 = \frac{-3-\sqrt{5}}{2}$. By $k \geq 1$, $n \geq 2$, and $|\alpha| \geq |\alpha_0|$ we have

$$|1 - 1/\alpha^k - 1/\alpha^{k+1} + 1/\alpha^{n+k}| > 0.4. \tag{12}$$

Since $-1 < \beta < 0$ we have $|\beta| < 1$ and $|\beta - 1| > 1$. Hence we get by (10), (11), and (12)

$$\frac{0.4|\alpha|^{n+k+1}}{|\alpha| + 1} < |F(\alpha)| = |F(\beta)| < \frac{4}{|\beta - 1|} < 4. \tag{13}$$

But (13) is a contradiction in view of $n \geq 2$, $k \geq 1$, and $|\alpha| \geq |\alpha_0| = \frac{3+\sqrt{5}}{2}$. Hence there are no balancing sequences with $A < 0, B < 0$ and $A^2 + 4B > 0$.

Case 4: $A < 0$ and $B > 0$.

Let us now consider the sequence $R_i = R(A, B, 0, 1)$ with $A < 0$ and $B > 0$. We also consider the corresponding sequence $Q_i := R(|A|, B, 0, 1)$. We clearly have $R_i = (-1)^{i-1} Q_i$ ($i \geq 1$) and thus $|R_i| = |Q_i| = Q_i$. Further, by induction on i it is easily seen that

$$Q_1 + Q_2 + \cdots + Q_{i-1} < Q_{i+1} \quad \text{for} \quad i = 2, 3, \ldots. \tag{14}$$

First we suppose $A \leq -2$. Now the absolute value of the left hand side of (1) is

$$|R_1 + \cdots + R_{n-1}| \leq Q_1 + \cdots + Q_{n-1}. \tag{15}$$

Further, by $Q_{i+1} = |A|Q_i + BQ_{i-1} \geq 2Q_i$ we have $Q_{i+1} - Q_i \geq Q_i$ for $i \in \mathbb{N}$ and the absolute value of the right hand side of (1) is

$$|R_{n+1} + \cdots + R_{n+k}| = |Q_{n+1} - Q_{n+2} + \cdots + (-1)^{k-1} Q_{n+k}|$$

and this is one of the following:

$$Q_{n+1}, \quad Q_{n+2} - Q_{n+1} \geq Q_{n+1}, \quad Q_{n+3} - Q_{n+2} + Q_{n+1} \geq Q_{n+1}, \ \ldots.$$

This together with (14) and (15) concludes the proof of Case 4 if $A \leq -2$.

Finally, if $A = -1$ then $Q_{i+1} - Q_i = BQ_{i-1}$ for $i \in \mathbb{N}$. Now the absolute value of the left hand side of (1) is

$$\begin{aligned} |R_1 + \cdots + R_{n-1}| &= |Q_{n-1} - Q_{n-2} + Q_{n-3} - Q_{n-4} + \cdots| \\ &\leq (Q_{n-1} - Q_{n-2}) + (Q_{n-3} - Q_{n-4}) + \cdots \\ &\leq B\,(Q_{n-3} + Q_{n-2} + \cdots + Q_1) < BQ_{n-1}. \end{aligned} \tag{16}$$

On the other hand, the right hand side of (1) is again one of the following:

$$Q_{n+1} > BQ_{n-1}, \quad Q_{n+2} - Q_{n+1} > BQ_{n-1}, \quad Q_{n+3} - Q_{n+2} + Q_{n+1} > BQ_{n-1}, \ \ldots.$$

This together with (14) and (16) concludes the proof of our Theorem 1.

5. Acknowledgement

We want to thank the referee for the comments concerning our manuscript. We found them to be most helpful and constructive.

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MSC2010: 11B37, 11B39, 11D99

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